# Lecture 36

PM in Bipartite Graphs, Error Reduction

# Perfect Matching in Bipartite Graphs

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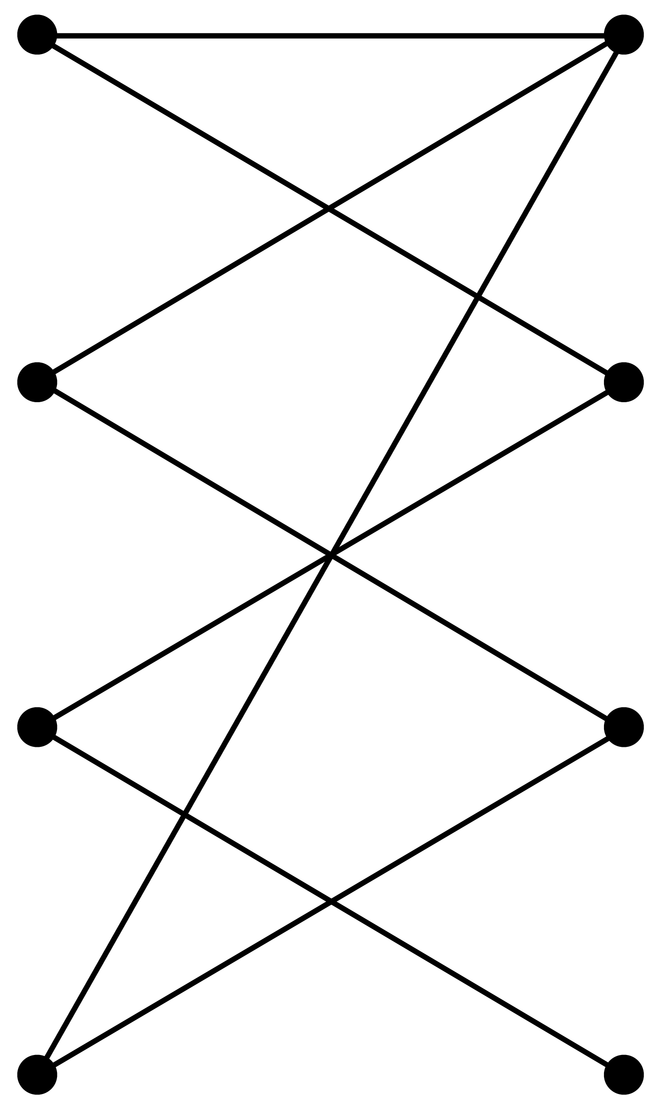

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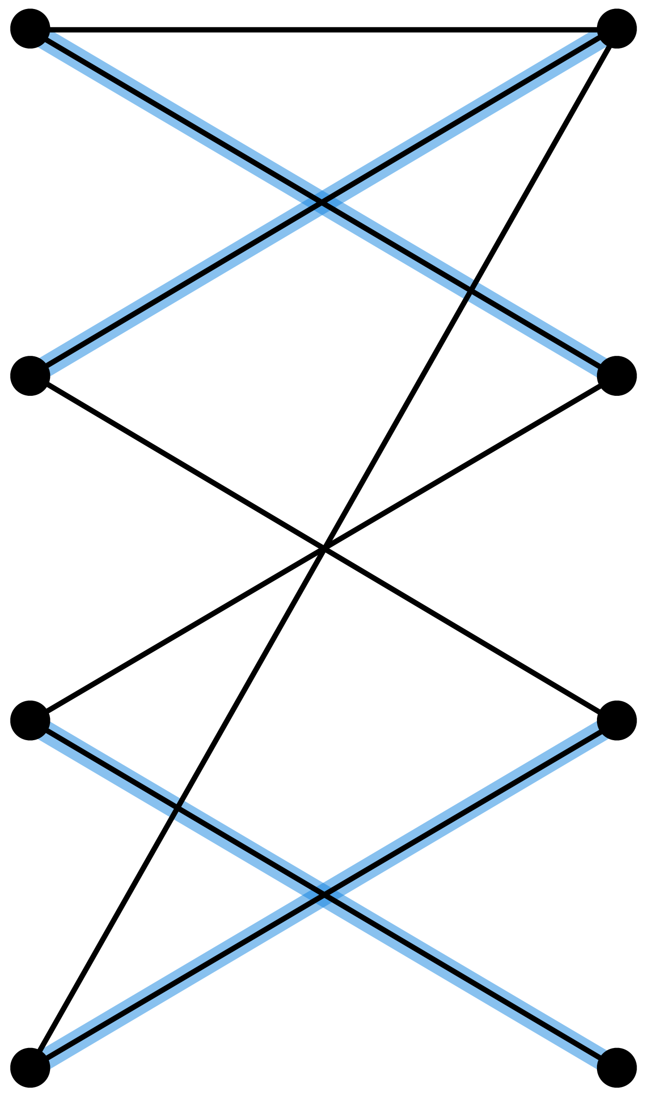

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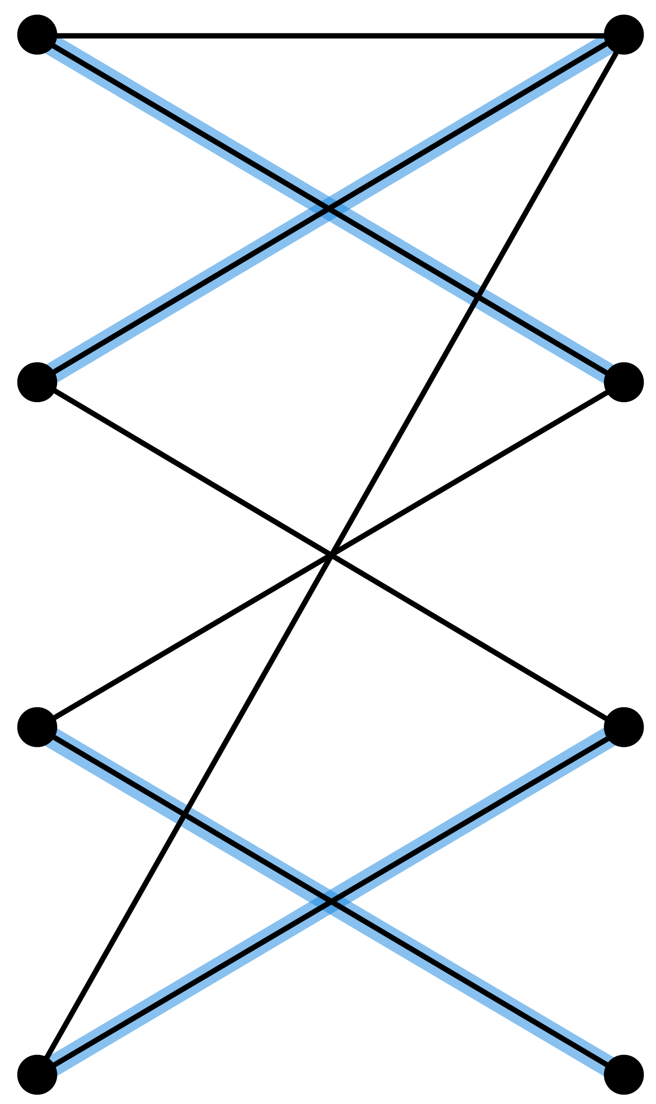

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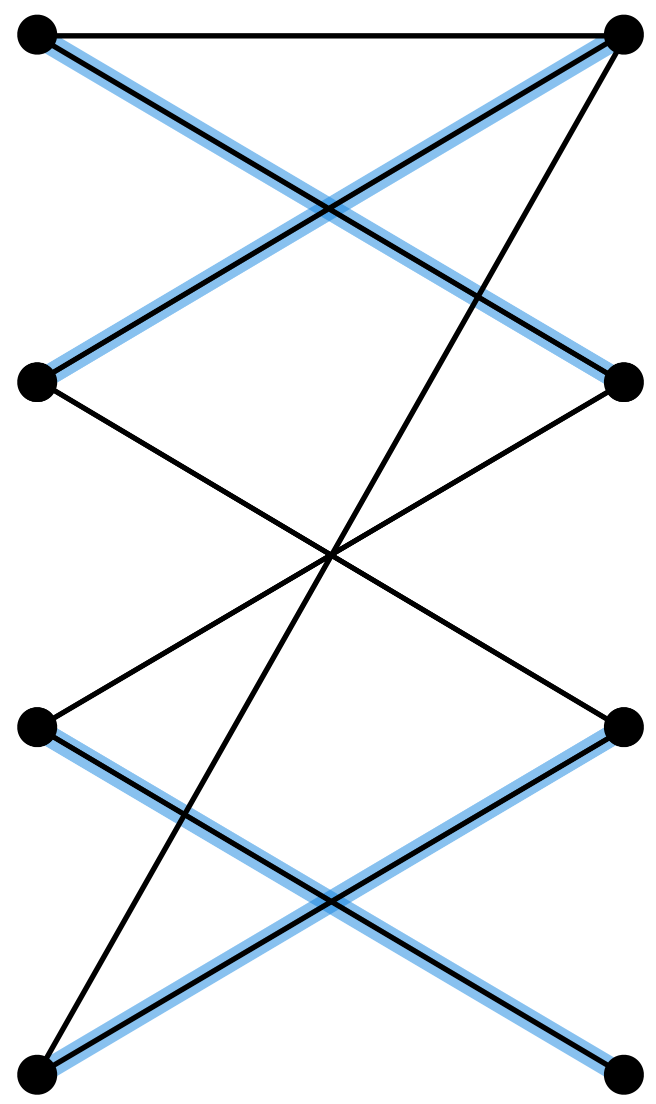

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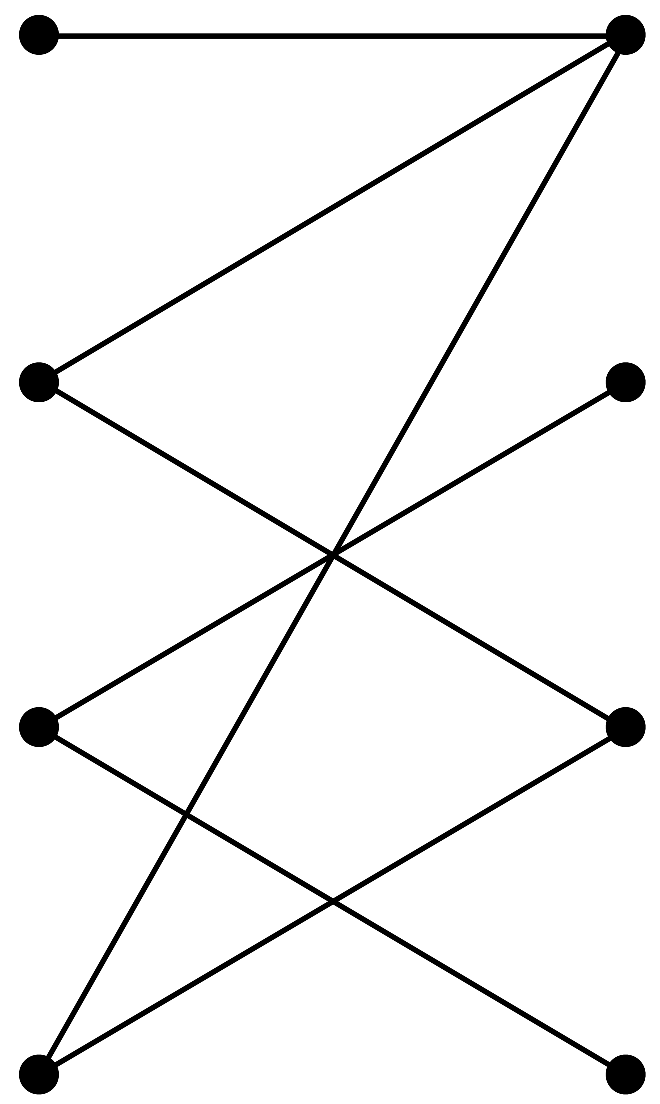

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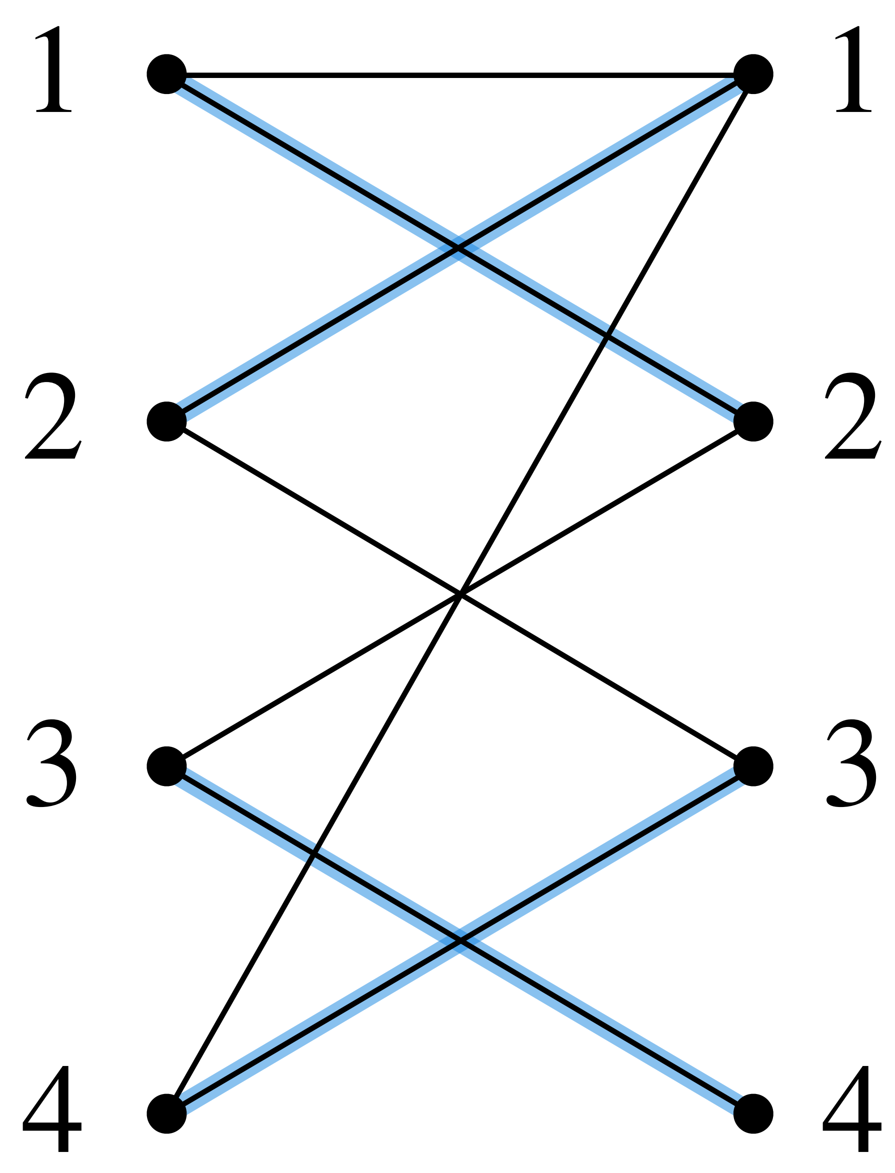


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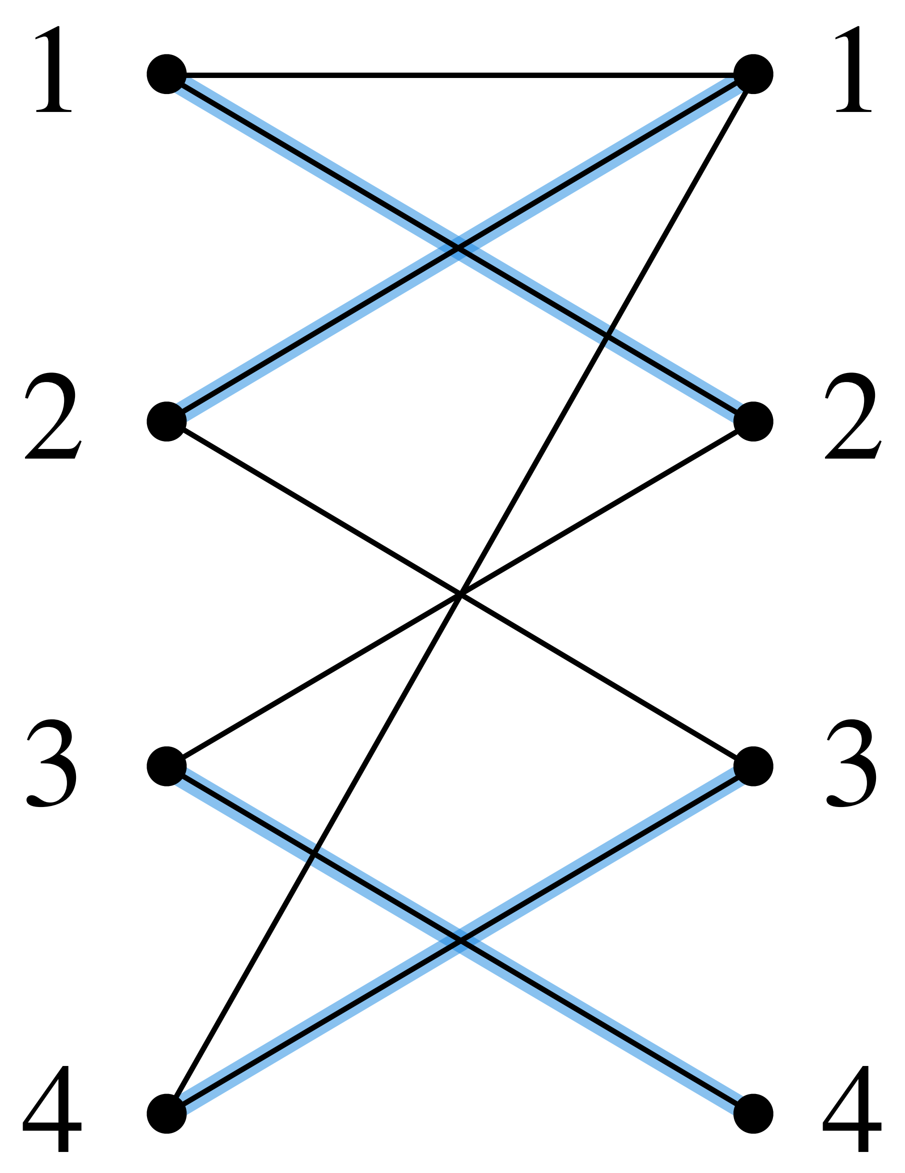


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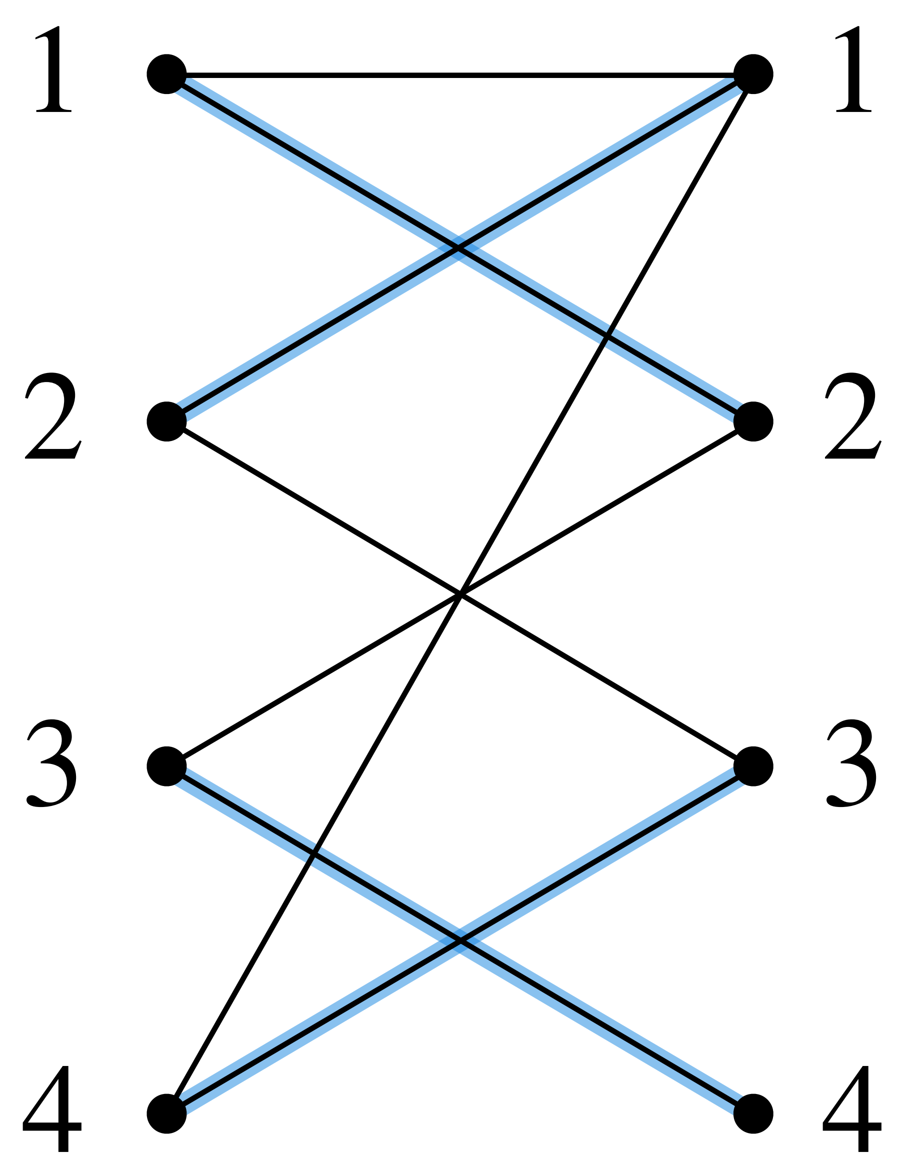


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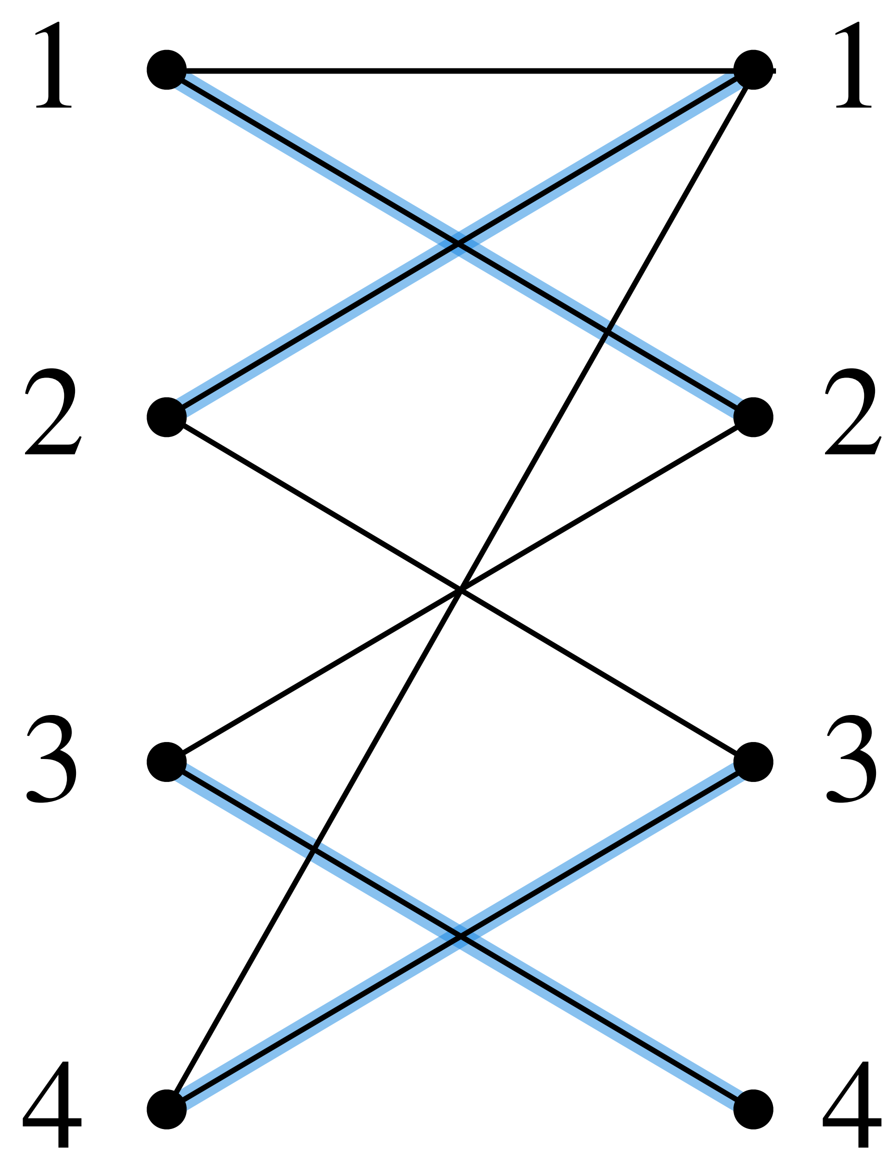

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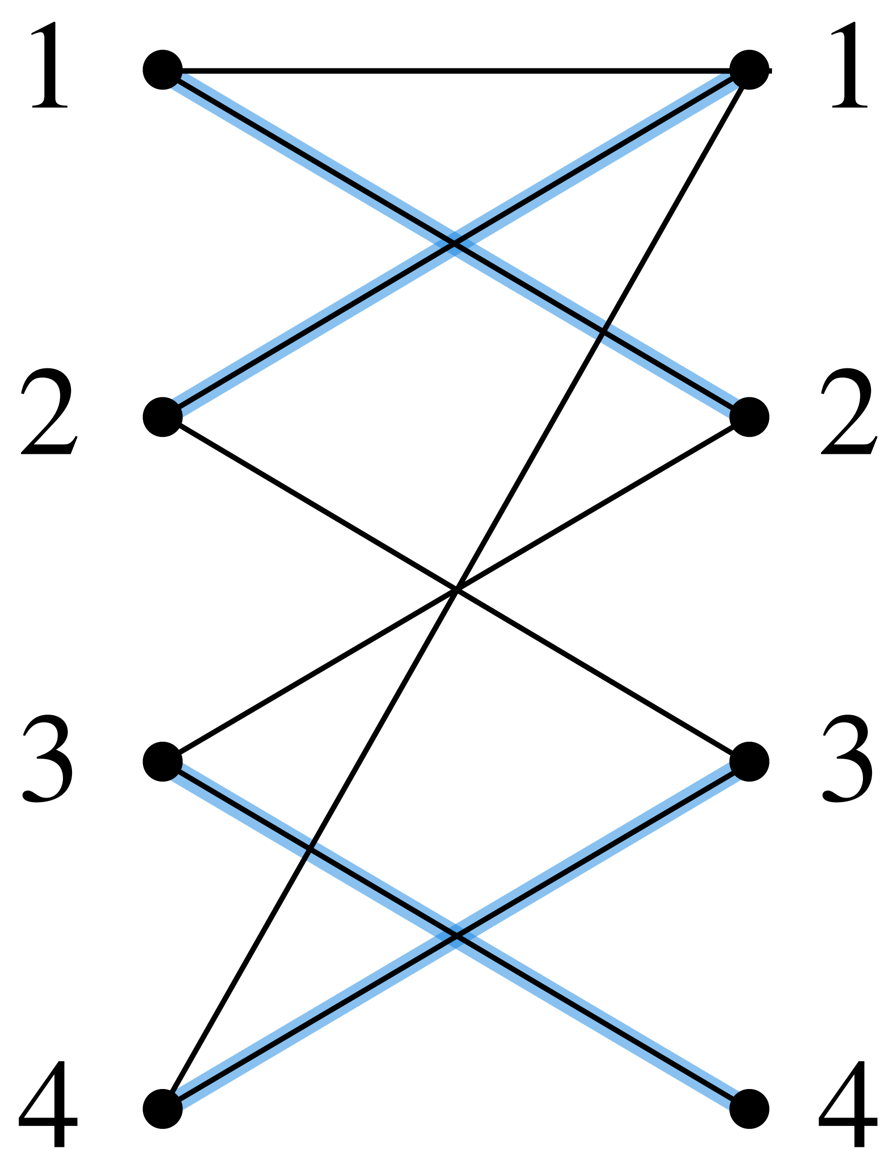


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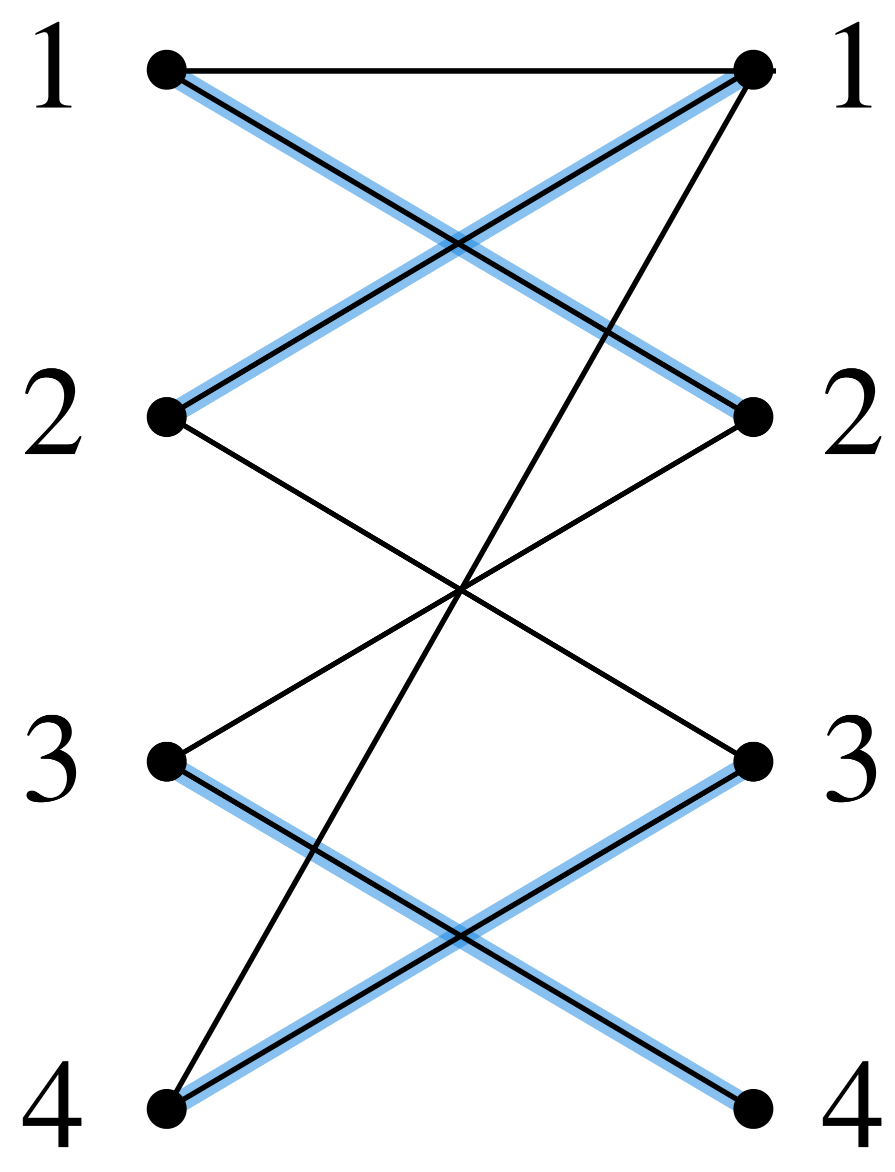


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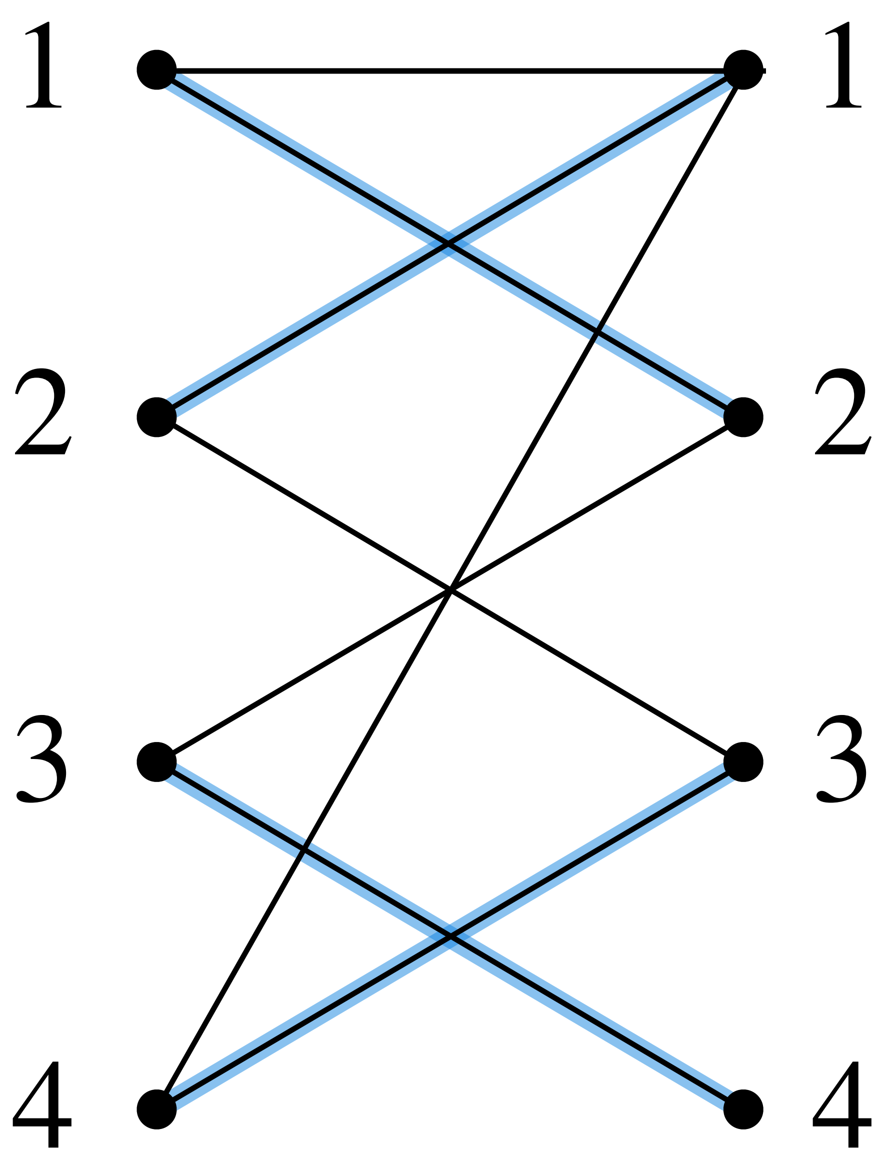


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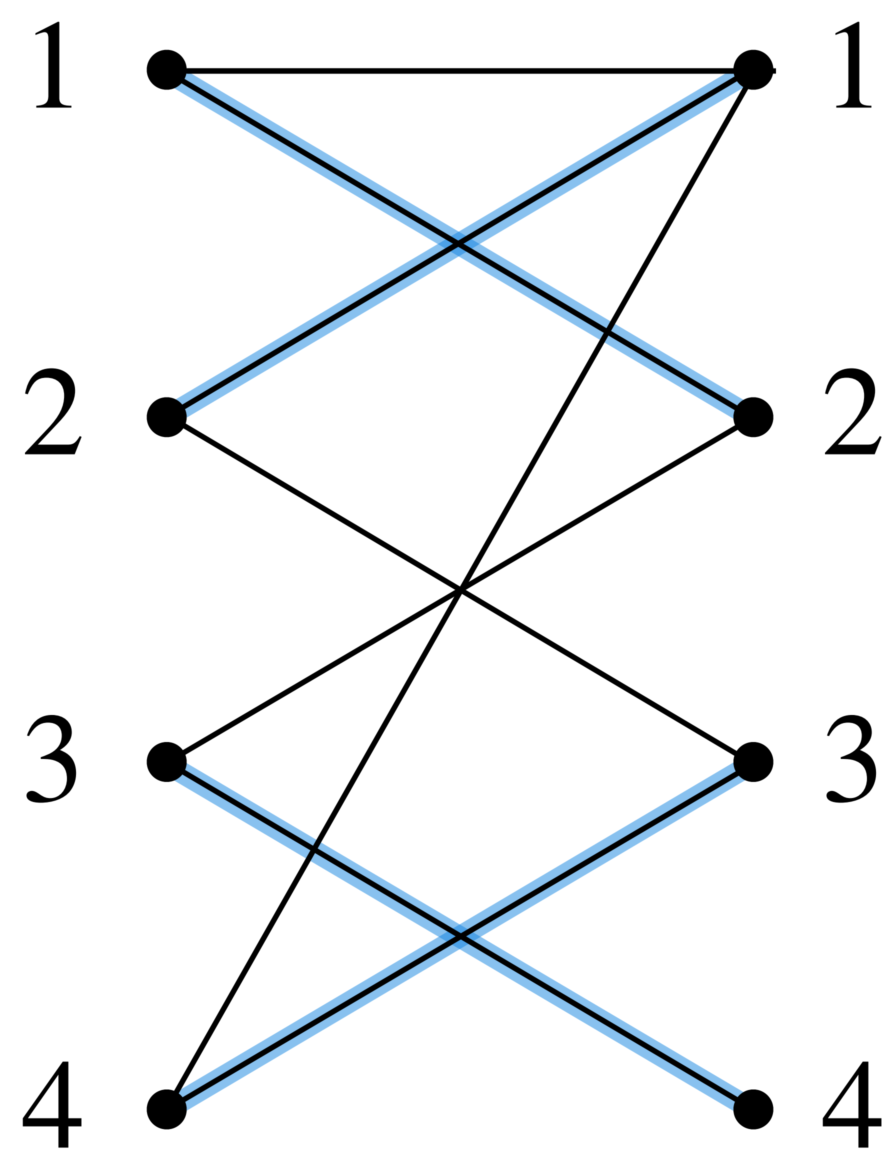


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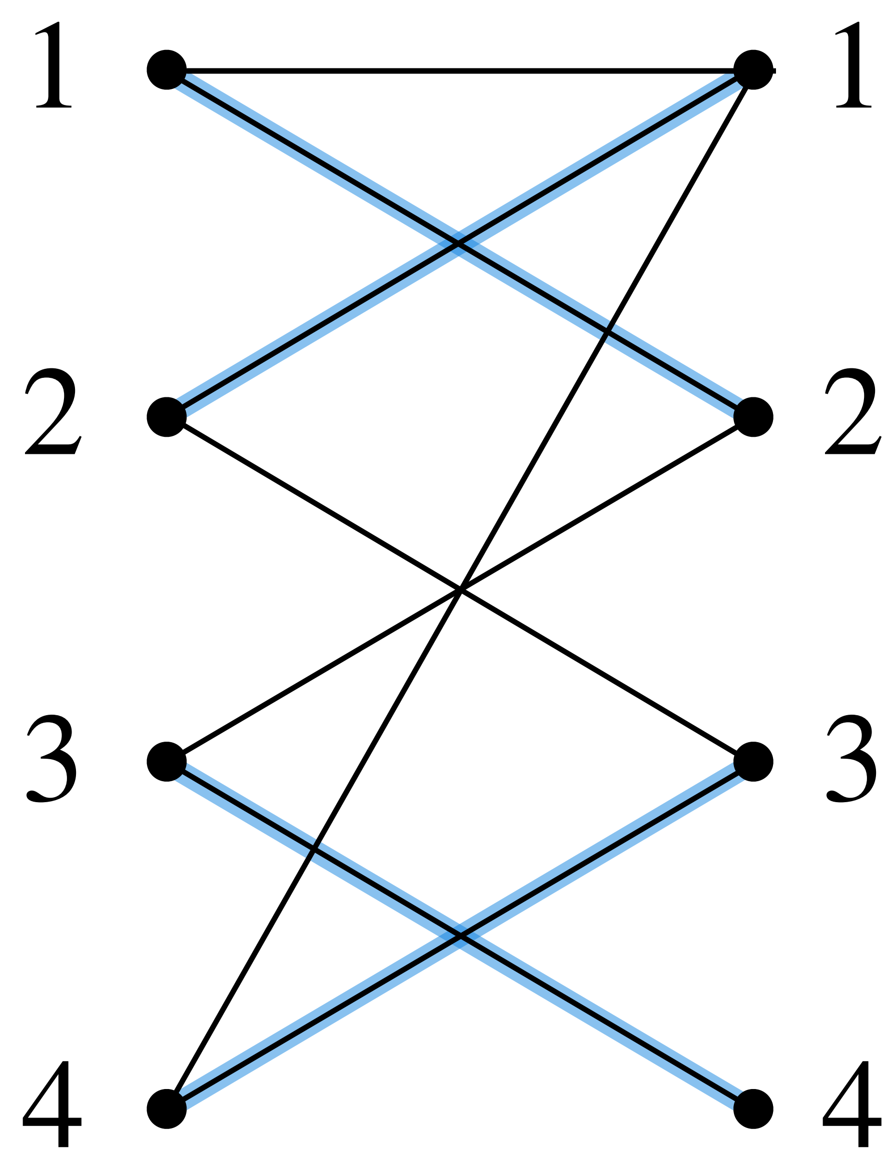


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**Corollary:** $G$ has a perfect matching iff $det(X)$ is a non-zero polynomial.

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**Theorem:** Let $L \in \text{BPP}_{\frac{1}{2}+}$. Then, for every constant $d > 1$, there exists a polytime PTM that

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# Reducing the Error Rate in BPP

**Definition:** For $c > 1$, a language $L$ is in $\mathsf{BPP}_{\frac{1}{2}+}$ if there exists a polytime PTM $M$ s.t. $\forall x \in \{0,1\}^*$,

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**Corollary:** $\mathsf{BPP}_{\frac{1}{2}+} = \mathsf{BPP}$.

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